INTRODUCTION OF WEAK SHOCK WAVES WITH RECTANGULAR MESHES IN PLATE

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In mechanical engineering, construction and other industries a significant part of the technological processes is associated with dynamic loads. Such activity can be single within the operating cycle as a package pulses or vibration. Each of these loads types can be of shock nature.

Obviously, for the actions of such stresses, especially impulse, stress concentration significantly differs from those occurring under conditions of quasi-static deformation. Research of dynamic problems for bodies with meshes was based on Laplace transform. In [1] using the method in the area of images of series the distribution of dynamic stresses in the plate with rectangular meshes from the effects of the shock load applied to the border was investigated. In [2, 3] solving the problem is reduced to the singular and regular integral equations.

In [4, 5] solving of dynamic problems in the case of antiplain deformation is performed by finite differences method by time.

In [6, 7] to study the dynamic stress state of plates with stress concentrators at antiplain strain the Fourier transform is used.

In [8] the technique of dynamic stressed state of plates with meshes at the steady-state oscillations is developed. This technique is based on joint use of boundary integral equations method and the apparatus of the complex variable theory. The numerical realization of the offered technique was made on the basis of a method of mechanical quadratures and collocation. Such approach was efficient when calculating of dynamic stress concentration in the plates weakened by meshes of the irregular shape.

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The aim is to develop an algorithm for calculating the dynamic stress state of plates with meshes for pulse loading in the form of a weak shock wave. This algorithm should be based on boundary integral equations method and apparatus of the complex variable theory.

Materials and methods. Let us consider an infinite isotropic elastic plate, weakened by a hole of any form (fig. 1). We will designate by $D$ the area which is occupied by a plate. Let $L$ — its limiting contour. We will carry an elastic plate to the Cartesian system of coordinates $Ox_1x_2$ which we will place in its gravity center.

According to [3], the equation of the movement of an isotropic plate in movements is written as:

$$
\left(c_1^2 - c_2^2\right)u_{i,j} + c_2^2u_{j,i} = \frac{\partial^2 u_i}{\partial t^2},
$$

(1)

where $u(x,t) = \{u_i(x,t), j = 1, 2\}$ — vector of arbitrary point movement $x = \{x_1, x_2\}$; $c_1 = \sqrt{\frac{(\lambda + \mu)}{\rho}}$, $c_2 = \sqrt{\frac{\mu}{\rho}}$ — speed expansion and shear of waves respectively; $\lambda, \mu$ — Lame constants; $\mathbf{b} = \{b_j\}$ — vector of mass forces; $(x_j)$, means for differentiation $x_j$; $t$ — time.

Using Fourier transform [9] to an equation (1) for the time variable $t$

$$
\tilde{f}(x, \omega) = \int_{-\infty}^{\infty} f(x, t)e^{-i\omega t}dt, \quad f(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(x, \omega)e^{i\omega t}d\omega
$$

(2)

gets equation:

$$
\left(c_1^2 - c_2^2\right)\tilde{u}_{i,j} - c_2^2\tilde{u}_{j,i} + \tilde{b}_j + \omega^2\tilde{u}_j = 0,
$$

(3)

which is equivalent to the equation steady-state oscillations with cyclic frequency $\omega$ [3]. The use of Fourier integral transform gives the opportunity to conduct research in the field of images, thereby isolate the effect of the time factor.

Research of dynamic stress state we will conduct in case when plane shock wave waves to the mesh in the plate, that is similarly to [10] defined by the potential:

$$
\phi(x, t) = \begin{cases} 
\phi_0 f(x/a - c_1 t / a), \ t \geq 0 \\
0, \ t < 0
\end{cases} \quad \psi(x, t) = 0,
$$

(4)

where $\phi_0$ — constant; $a$ — characteristic size.

Applying to the presentation (4) the integral Fourier transform (2), we get:

$$
\tilde{\phi}(x, \omega) = \tilde{\phi}_0 \frac{a}{c_1} \tilde{f} \left( \frac{\omega a}{c_1} \right) e^{-i\omega x} / c_1 = \tilde{\phi}_0(\omega)e^{-i\omega x} \quad \tilde{\psi}(x, \omega) = 0,
$$

(5)

where $\tilde{\phi}_0(\omega) = \tilde{\phi}_0 \frac{a}{c_1} \int_{-\infty}^{\infty} \tilde{f}(\tau)e^{-i\omega \tau}d\tau$.

Boundary conditions of the problem in the field of Fourier-images we will write as [3]:

$$
\tilde{\sigma}_i |_{x_1} = \tilde{\sigma}(x, \omega), \ \tilde{\tau}_i |_{x_2} = \tilde{\tau}(x, \omega).
$$

(6)

For the case of the first main problem the potential image of general solution for displacements we will choose as [10]:

![Fig. 1. Scheme of loading of a plate]
where $p_1, p_2$ — unknown complex potential functions;
$L$ — border of plate area.
Integration along the border is held by the variables $x_0^i, x_0^j$, and the $\mathbf{x}^0 = \{x_0^i, x_0^j\}$. Expressions for images of functions $U^\star$ are chosen accounting the Sommerfeld conditions [10] as [3]:

$$U^\star_j = \frac{1}{2\pi i} (\psi \cdot \delta_j - \chi \cdot r_j \cdot r_j),$$  \hspace{1cm} (8)

where $\psi = K_0(k_j r) + \frac{1}{k_j r} \left( K_i(k_j r) - \frac{c_i}{c_j} K_i(k_i r) \right)$, $\chi = K_2(k_j r) - \frac{c_i^2}{c_j^2} K_2(k_i r)$;

$k_j = \frac{i0}{c_j},$ $j = 1, 2$ — wave number;

$K_m(r), m = 0, 1, 2$ — modified Bessel functions of the third kind (or Macdonald functions);

$r = \sqrt{(x_0^i - x_0^j)^2 + (x_1^i - x_1^j)^2}$ — distance.

Satisfying the boundary conditions (6), we determine the unknown function on border $p_1, p_2$, calculating the stresses in the plate by formulas [10]:

$$\sigma_n = \frac{\sigma_{i1} + \sigma_{i2}}{2} + \frac{1}{2} \left( e^{-2\alpha} \left( \frac{\sigma_{i1} + \sigma_{i2}}{2} + i\sigma_{i2} \right) + e^{2\alpha} \left( \frac{\sigma_{i1} + \sigma_{i2}}{2} - i\sigma_{i2} \right) \right);$$

$$\tau_n = i \left( e^{-2\alpha} \left( \frac{\sigma_{i1} + \sigma_{i2}}{2} - i\sigma_{i2} \right) - e^{2\alpha} \left( \frac{\sigma_{i1} + \sigma_{i2}}{2} + i\sigma_{i2} \right) \right),$$  \hspace{1cm} (9)

where $\alpha$ — the angle between the normal line $\mathbf{n}$ the plate and the axis $Ox_1$.

Substituting dependences (7) taking into account the expressions for functions of influence (8) in formulas (9), we will obtain a representation like:

$$\sigma_n = \int \int f_i(z, \xi) q d\xi \xi + \int \int f_j(z, \xi) q d\xi \xi; \hspace{1cm} \tau_n = \int \int g_i(z, \xi) q d\xi \xi + \int \int g_j(z, \xi) q d\xi \xi,$$  \hspace{1cm} (10)

where $q = i \cdot p \cdot ds / d\xi$ — unknown complex functions;

$\xi = x_0^i + ix_0^j, z = x_1 + ix_1, p = p_1 + ip_2;

f_i, g_i, i = 1, 2$ — known function containing Bessel functions of the third kind:

$$f_i = -\frac{i}{4\pi i} \left( (z - \xi) F - \left( \frac{dz}{dz} (z - \xi)^2 G_i + \frac{d\xi}{dz} (z - \xi) G_i \right) \right);$$

$$f_j = \frac{i}{4\pi i} \left( (z - \xi) F - \left( \frac{dz}{dz} (z - \xi) G_i + \frac{d\xi}{dz} (z - \xi) G_i \right) \right);$$

$$g_i = -\frac{1}{4\pi i} \left( \frac{dz}{dz} (z - \xi) G_i - \frac{d\xi}{dz} (z - \xi) G_i \right);$$

$$g_j = \frac{1}{4\pi i} \left( \frac{dz}{dz} (z - \xi) G_i - \frac{d\xi}{dz} (z - \xi) G_i \right);$$

$$F = \psi' - \chi'; G_i = \psi' - \frac{\chi'}{2}, G_j = \frac{\psi'}{r} - \frac{\chi'}{2}.$$
The integrands \( f_i, g_i, i=1, 2 \) at small values of the argument are irregular. Let set their features, using the asymptotic expressions for the Bessel functions of the third kind [12]:

\[
K_\nu(r) = \ln 2 - \ln 2 - \gamma + O(r^\nu); \quad K_\nu(r) = \frac{1}{r} + \frac{1}{2} \left( -\ln 2 + \ln r + \gamma - \frac{1}{2} \right) + O(r^\nu); \quad K_\nu(r) = \frac{2}{r^2} - \frac{1}{2} + O(r^\nu),
\]

where \( \gamma \) — Euler’s constant.

We obtain, that at \( r \to 0 \) functions \( \psi \) and \( \chi \) have features of type:

\[
\psi = -\frac{1}{2} \ln r (1 + \kappa) + O(r^\nu); \quad \chi = -\frac{1}{2} (1 - \kappa) + O(r^\nu), \quad \kappa = \left( \frac{c_1}{c_2} \right)^2.
\]

Besides, functions \( F, G_i, G_j \) have features of type:

\[
F = -\frac{\kappa}{r} + O(r^\nu); \quad G_i = -\frac{1 + \kappa}{2r} + O(r^\nu); \quad G_j = -\frac{1 - \kappa}{2r} + O(r^\nu).
\]

In the case of plane stressed state \( \kappa = (1 - \nu)/2, \nu \) — Poisson’s ratio.

Considering the above mentioned features of integrand functions, the representation for definition of the functions \( f_i, g_i, \, i=1, 2 \) can be written as:

\[
f_1 = -i \frac{1}{4\pi \mu} \left( \frac{\kappa}{z - \xi} + \frac{1}{2} \left( 1 - \kappa \right) \frac{dz}{d(\xi - z)} \right) + f_1^*(z, \xi);
\]

\[
f_2 = i \frac{1}{4\pi \mu} \left( \frac{\kappa}{\xi - z} + \frac{1}{2} \left( 1 + \kappa \right) \frac{dz}{d(\xi - z)} \right) + f_2^*(z, \xi);
\]

\[
g_1 = \frac{1}{8\pi \mu} \left( 1 - \kappa \right) \frac{dz}{d(\xi - z)} - \frac{dz}{d(\xi - z)} \right) + g_1^*(z, \xi);
\]

\[
g_2 = -\frac{1}{8\pi \mu} \left( 1 + \kappa \right) \frac{dz}{d(\xi - z)} - \frac{dz}{d(\xi - z)} \right) + g_2^*(z, \xi),
\]

where the functions \( f_i^*, g_i^*, i=1, 2 \) are regular functions.

We apply the Sokhotski-Plemelj formula [10] at the boundary transition in dependences (10) with consideration of the obtained representations. As a result of these changes we obtain the system of integral equations to find the unknowns at functions border, \( q, q \):

\[
\begin{aligned}
\text{Re} \frac{q}{2} &= \text{Im} \left( A_1 \int \frac{zdz}{\xi - z} + A_2 \int \frac{zd\xi}{\xi - z} + A_3 \int \frac{zd\xi}{(\xi - z)^2} \right) + \int f_1^*(z, \xi) q d\xi + \int f_2^*(z, \xi) q d\xi = \sigma; \\
\text{Im} \frac{q}{2} &= -\text{Re} \left( A_1 \int \frac{zdz}{\xi - z} + A_2 \int \frac{zd\xi}{\xi - z} + A_3 \int \frac{zd\xi}{(\xi - z)^2} \right) + \int g_1^*(z, \xi) q d\xi + \int g_2^*(z, \xi) q d\xi = \tau;
\end{aligned}
\]

where \( A_1 = \frac{1 + \nu}{4\pi \nu}, \quad A_2 = \frac{3 - \nu}{4\pi}, \quad A_3 = \frac{1 + \nu}{4\pi} \) — the constants defined for a case of flat stressed state;

\( \sigma, \tau \) — known functions that are based on representation (5);

\( f_i^*, g_i^*, i=1, 2 \) — known regular functions.

Obtained system of integral equations (11) we solve numerically, using method of mechanical quadratures and collocation [12]. For integrals with Cauchy type feature we apply the clarified quadrature formulas [12]. We obtain a system of linear algebraic equations of the form:
\[
\begin{align*}
\bar{g}_{\alpha} = & \frac{q_{\alpha} + \bar{q}_{\alpha}}{4} + \sum_{n=1}^{N} f_{\alpha}(z_{\alpha}, \xi_{\alpha}) q_{\alpha} s_{\alpha}^n + \sum_{n=1}^{N} f_{\alpha}(z_{\alpha}, \xi_{\alpha}) \bar{q}_{\alpha} s_{\alpha}^n = \Phi_{\alpha}^s; \\
\bar{g}_{\beta} = & \frac{q_{\beta} + \bar{q}_{\beta}}{4i} + \sum_{n=1}^{N} g_{\beta}(z_{\beta}, \xi_{\beta}) q_{\beta} s_{\beta}^n + \sum_{n=1}^{N} g_{\beta}(z_{\beta}, \xi_{\beta}) \bar{q}_{\beta} s_{\beta}^n = \Phi_{\beta}^s;
\end{align*}
\]

(12)

where \( \xi_{\alpha} = g(\varphi_{\alpha}) \), \( z_{\alpha} = g(\varphi_{\alpha}) \), \( \varphi_{\alpha} = k \alpha + \frac{h}{N}, \) \( h = \frac{2\pi}{N} \),

\[ g(\varphi) \] — parametric boundary setting \( L \);

\[ \Phi_{\alpha}, \Phi_{\beta} \] — \( \hat{\sigma}(\varphi_{\alpha}), \hat{\tau}(\varphi_{\beta}) \);

\( N \) — number of nodal points.

The calculation of circular stresses in plate held numerically based on dependencies obtained from the formulas [10] by allocating irregular components and use of Sokhotski-Plemelj formula at the boundary transition:

\[
\bar{\sigma}_{\alpha} = \nu \text{Re} \left\{ \frac{q_{\alpha}}{2} - \text{Im} \left( A \int_{\xi} \frac{q d\xi}{\zeta - \xi} d\xi + A \int_{\xi} \frac{q d\xi}{\zeta - \xi} d\xi \right) \right\} + \int_{\xi} y_{\alpha}(z, \xi) q d\xi + \int_{\xi} y_{\beta}(z, \xi) \bar{q} d\xi;
\]

(13)

where \( y_{\alpha}(z, \xi), y_{\beta}(z, \xi) \) — functions that do not have features.

Definition of originals of obtained on the basis of formulas (13) stresses was conducted using inverse Fourier transform [9] according to dependences:

\[
\sigma_{\alpha}(x, t) = \frac{1}{2\pi} \int_{\omega=-\infty}^{\omega=\infty} \tilde{\sigma}_{\alpha}(x, \omega) e^{i\omega t} d\omega,
\]

(14)

which at numerical calculation of circular stresses can be implemented based on the discrete Fourier transform, which proved its effectiveness at researches of the dynamic stress condition at antiplane strain [6].

**Results.** Based on the proposed algorithm will explore the dynamic stress concentration in plate weakened by rectangular mesh with aspect ratio of 3.2. To describe the border of the plate mesh we use the presentation as a series used in the conformal reflection of a circle of radius \( a \) to given region [13]:

\[
g(\varphi) = a \left( e^{-n} + e^{i\omega} - \frac{3e^{i\omega}}{8} - \frac{5e^{i\omega}}{780} + \frac{112645}{11264} \right),
\]

limiting the number of 11 members.

Numerical calculations we will perform for case of modulation of pulse by time in the form of a weak shock wave [6,7]:

\[
f(\tau) = p, \tau^n e^{-\omega \tau}, \quad \tau > 0, n, \geq 0,
\]

where \( \hat{f}(\omega) = p, (n, l)(\alpha, + i\omega)^{-n-1} \).

In the calculations such values are taken, similar to [6, 7], \( p_{\omega} = 185; n_{\omega} = 2; \alpha_{\omega} = 10; \omega = 1 \).

Research carried out for the interval of dimensionless time parameter \( T \epsilon [0, 8a/c] \).

As calculated by the formula (14) values of circular stresses \( \sigma \theta \) images in general are complex, then Fig. 1 shows the dependence of the changes over time of real and imaginary values of dynamic circular stresses: \( \sigma_{\alpha}^{\theta} = \text{Re}(\sigma_{\alpha}), \sigma_{\beta}^{\theta} = \text{Im}(\sigma_{\beta}) \), calculated at \( A \) point of a border. The calculation results shown in Figure 1 were made at a value of Poisson's ratio \( \nu = 0.3 \) and \( N = 300 \) of nodal points on the border of the plate.

Based on Fig. 1 it can be analyzed the distribution of weak shock waves in the plate, that come to the mesh border with time. Increasing of the values of dynamic stresses components \( \sigma_{\alpha}^{\theta} \) and \( \sigma_{\beta}^{\theta} \), that starts from \( T = 0.22a/c \), corresponds to "coming" to "end" of mesh of disturbing impact \( \varphi(x, \tau) \).
Further growth of stresses in the plate is associated with the spreading of reflected and rereflected pulses from the edges of the mesh. Fig. 1 shows that the intensity of the shock wave rapidly decreases at reflection from the borders of the mesh. The main influence on the dynamic plastic stress state has the main wave and the first reflected wave from the right border.

For a detailed study of unsteady process at the plate over time we build time sections of dynamic stresses distribution along the border of the mesh. The calculation results are shown in Fig. 2 for the time interval $T \in [0; 1.2a/c_1]$ with step $\Delta T = 0.2a/c_1$ (curve 1…7).

Fig. 2 shows that under the influence of a weak shock wave the maximum dynamic stresses on the border of the mesh occur in the neighborhood of point A. Further distribution of stresses in the plate essentially depends on defragged waves from its mesh. Numerical calculations showed that the main shock of wave loading will reach the border of the mesh and reflect from it at $T = 0.2a/c_1$. Having $T = 1.6a/c_1$ the shock wave will reflect from the left side of the border. During the spread of reflected waves in a plate it is seen a significant increase of dynamic stress in angular points.

Having calculated the discrete time representation for movements that are similar to the formula (14) we can analyze a complete picture of unsteady flow wave process at the border of the mesh plate.

**Conclusions.** The use of integral Fourier transform over time and inverse discrete Fourier transform makes it possible to reduce the dynamic problem of plane deformation to solving the finite number of problems on steady-state oscillations at fixed values of cyclic frequencies. The advantage of the proposed approach is the setting for the incident wave for images area as (5), which makes it possible to calculate using the formula (14) the values of circular stresses for time moments $T_k \in [0, T]$, and not just half of them, as it does the calculations based on the discrete Fourier transform [9]. Thus the developed approach based on the method of boundary integral equations and the theory of functions of complex variables in field of Fourier-images makes it possible to calculate the time dependence of dynamic stresses in bodies with meshes for the actions of impulsive dynamic loads.

Література


References


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